

# CRITICAL SETS OF PROPER HOLOMORPHIC MAPPINGS

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**ABSTRACT.** It is shown that if a proper holomorphic map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$ ,  $1 < n \leq N$ , sends a pseudoconvex real analytic hypersurface of finite type into another such hypersurface, then any  $n - 1$  dimensional component of the critical locus of  $f$  intersects both sides of  $M$ . We apply this result to the problem of boundary regularity of proper holomorphic mappings between bounded domains in  $\mathbb{C}^n$ .

## 1. INTRODUCTION AND MAIN RESULTS

The goal of this article is to prove the following theorem that describes geometry of the critical set of a proper holomorphic map between real analytic hypersurfaces.

**Theorem 1.** *Let  $D \subset \mathbb{C}^n$ ,  $D' \subset \mathbb{C}^N$ ,  $2 \leq n \leq N$ , be domains and  $f : D \rightarrow D'$  be a proper holomorphic map that extends holomorphically to a neighbourhood  $U \subset \mathbb{C}^n$  of a point  $a \in \partial D$ . Suppose that  $\partial D \cap U$  and  $\partial D' \cap U'$  are smooth real analytic pseudoconvex hypersurfaces of finite type, where  $U' \subset \mathbb{C}^N$  is a neighbourhood of  $f(a) \in \partial D'$ . Let  $E$  be an irreducible  $(n-1)$ -dimensional component of the critical set of  $f$  in  $U$  with  $a \in E$ . Then  $E \cap (D \cap U) \neq \emptyset$ .*

We note that the neighbourhood  $U \ni a$  in Theorem 1 for which  $E \cap (D \cap U) \neq \emptyset$  is arbitrarily small. In this case we say that  $E$  enters the domain  $D$  at the point  $a$ .

We apply Theorem 1 to the study of the old conjecture that a proper holomorphic map  $f : D \rightarrow D'$  between bounded domains in  $\mathbb{C}^n$  with real analytic boundaries extends holomorphically to a neighbourhood of the closure of  $D$ . The history of this conjecture began in the 70-ties when it was proved for strictly pseudoconvex domains by Lewy [17] and Pinchuk [18]. The conjecture has been studied by many authors but still remains open in full generality. However, it has been proved in the following considerable special cases:

- (1)  $D, D'$  are pseudoconvex,  $n \geq 2$  (Diederich-Fornaess [9], Baouendi-Rothchild [1]);
- (2)  $n = 2$  (Diederich-Pinchuk [10]);
- (3)  $f$  is continuous in the closure of  $D$ ,  $n \geq 2$  (Diederich-Pinchuk [12]).

The proofs of these results consist of two major steps. Step 1 is to show that  $f$  extends as a holomorphic correspondence to a neighbourhood of the closure of  $D$ . Step 2 is to prove that this correspondence is, in fact, a holomorphic map. The main method for step 1 is the multidimensional reflection principle, based on the technique of Segre varieties. For a survey on the subject we refer the reader to [14]. Except the case  $n = 2$ , step 1 was realized so far only under additional assumption of some a priori boundary regularity of  $f$ . In particular, in [12] it was proved provided that  $f \in C(\overline{D})$ . We also note that continuous extension of  $f$  to  $\overline{D}$  was proved in pseudoconvex case by Diederich-Fornaess [8]. Step 2 is essentially the following result.

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**Theorem 2.** *Let  $D, D' \subset \mathbb{C}^n$ ,  $n \geq 2$ , be bounded domains with real-analytic boundaries and  $f : D \rightarrow D'$  be a proper holomorphic map that extends as a holomorphic correspondence to a neighbourhood of  $\overline{D}$ . Then  $f$  extends holomorphically to a (possibly smaller) neighbourhood of  $\overline{D}$ .*

Theorem 2 and its generalizations have been proved in [11], [13], [19] and strongly rely on the proof in the case when both domains are pseudoconvex. The key for proving Theorem 2 in the pseudoconvex case is the  $C^\infty$ -smooth extension of  $f$  to the closure of  $D$  (see, for instance, [2, 3]). However, the existing proof of the  $C^\infty$  extension is based on very technical and complicated subelliptic estimates for  $\bar{\partial}$ -Neumann operator [16]. Here we use Theorem 1 to present a more elementary self-contained proof of Theorem 2 in general situation. This allows us to simplify previous proofs of the results discussed above by avoiding the use of sophisticated  $\bar{\partial}$ -machinery. In fact, while Theorem 2 is stated for simplicity as a global result, we prove a local version of it.

## 2. BACKGROUND: SEGRE VARIETIES, THE SEGRE MAP AND ITS CRITICAL LOCUS.

Let  $M$  be a smooth real analytic hypersurface in  $\mathbb{C}^n$  passing through the origin. In a suitable coordinate system we may assume that it is given by a defining function

$$\rho(z, \bar{z}) = z_n + \bar{z}_n + \sum_{|j|, |k| > 0} a_{jk}(y_n)' z^j \bar{z}^k,$$

where  $'z = (z_1, \dots, z_{n-1})$ . By the Implicit Function Theorem, the complexified equation  $\rho(z, \bar{w}) = 0$  can be solved for  $z_n$ :

$$z_n = -\bar{w}_n + \sum_k \overline{\lambda_k(w)'} z^k, \quad k = (k_1, \dots, k_{n-1}). \quad (1)$$

The Segre varieties are defined as  $Q_w = \{z : \rho(z, \bar{w}) = 0\}$ , and  $M$  is called *essentially finite* at zero, if the Segre map  $\lambda : w \rightarrow Q_w$  is finite in a neighbourhood of the origin. The Segre map can be identified with the holomorphic map  $\lambda(w) = \{\lambda_k(w)\}$ , where  $\lambda_k$  are the components of the sum in (1). In fact, if  $M$  is essentially finite at zero, then there exists  $m > 0$  such that

$$Q_w = Q_{\tilde{w}} \iff \lambda_k(w) = \lambda_k(\tilde{w}), \quad |k| \leq m,$$

see [5] or [14] for the proof. Hence, we may identify the Segre map  $\lambda$  with a holomorphic map from a neighbourhood of the origin in  $\mathbb{C}^n$  into  $\mathbb{C}^N$ , for some  $N > 0$ , given by

$$\lambda(w) = \{\lambda_k(w), \quad |k| \leq m\}.$$

A smooth real hypersurface  $M$  is of *finite type* (in the sense of D'Angelo) at a point  $p \in M$ , if the order of contact of  $M$  with any one-dimensional complex analytic set passing through  $p$  is bounded above. If  $M$  is real analytic, then  $M$  is of finite type at  $p$  if and only if there does not exist a germ at  $p$  of a positive dimensional analytic set contained in  $M$ . In particular, this means that  $M$  is essentially finite near  $p$ , and so the Segre map is finite.

## 3. PROOF OF THEOREM 1

*Proof of Theorem 1.* Without loss of generality we may assume that  $a = 0$ ,  $f(0) = 0'$ , and  $f(U) \subset U'$ . Clearly,  $f(D \cap U) \subset D' \cap U'$  and  $f(\partial D \cap U) \subset \partial D' \cap U'$ . By the result of Diederich and Fornaess [6], for any  $\varepsilon > 0$  in a sufficiently small neighbourhood  $U'$  of the origin the hypersurface  $\partial D' \cap U'$  admits a defining function  $\rho' \in C^2(U')$  such that  $\phi' := -(-\rho')^{1-\varepsilon}$  is a plurisubharmonic

function on  $D' \cap U'$ . It follows that  $\phi' \circ f$  is a negative plurisubharmonic function in  $D \cap U$ , and so by the Hopf lemma there exists a constant  $C > 0$  such that

$$|\phi' \circ f(z)| \geq C \text{dist}(z, \partial D), \quad z \in \partial D \cap U. \quad (2)$$

Throughout the paper  $\text{dist}(\cdot, \cdot)$  denotes the usual Euclidean distance between sets in a Euclidean space. We may assume that complex tangents to  $\partial D$  and  $\partial D'$  at 0 and  $0'$  are given respectively by  $\{z_n = 0\}$  and  $\{z_N = 0\}$ . Then it follows from (2) that

$$\frac{\partial f_N}{\partial z_n}(0) \neq 0. \quad (3)$$

Indeed, if otherwise  $\frac{\partial f_N}{\partial z_n}(0) = 0$ , then  $f_N(z) = O(|z|^2)$ , and since  $\rho'(z') = 2x'_N + O(|z'|^2)$ , we obtain

$$|\phi' \circ f(z)| \leq c_1 |z|^{2(1-\varepsilon)},$$

which contradicts (2) for  $\varepsilon < 1/2$ . In particular, we conclude that the map  $f$  extends to  $U$  as a proper holomorphic map. This can be seen as follows: (3) implies that  $f(U \setminus D) \subset U'$ , and therefore,  $f^{-1}(\partial D \cap U') \subset \partial D$ . Since  $\partial D$  is of finite type, the set  $f^{-1}(0')$  is discrete, and, after shrinking if necessary the neighbourhood  $U$ , we may assume that the map  $f$  is proper in  $U$ .

By Remmert's proper mapping theorem  $E' = f(E) \subset U'$  is an irreducible analytic set of dimension  $n - 1$ . To illustrate the idea of the proof of the theorem consider first the simple case when  $E$  and  $E'$  are complex manifolds. Arguing by contradiction suppose that  $E \cap (D \cap U) = \emptyset$  for arbitrarily small  $U$ . Then  $E$  is tangent to  $\partial D$  at the origin. Since  $E' \cap (D' \cap U')$  is also empty, the manifold  $E'$  is tangent to  $\partial D'$  at  $0'$ . After an additional local biholomorphic change of coordinates we may assume that  $E = \{z_n = 0\}$  and  $E' = \{z'_N = 0\}$ . Let  $z = (\tilde{z}, z_n)$ ,  $z' = (\tilde{z}', z'_N)$ , and  $f = (\tilde{f}, f_N)$ . Then the restriction  $f|_E$  is given by  $\tilde{z}' = \tilde{f}(\tilde{z}, 0)$ . Since  $f$  is proper at the origin,  $f|_E$  is also proper at 0, and therefore the rank of the Jacobian matrix  $\frac{\partial \tilde{f}}{\partial \tilde{z}}(\tilde{z}, 0)$  is equal to  $n - 1$  on a dense subset  $E_1 \subset E$ . On the other hand,  $\text{rank} \frac{\partial f}{\partial z} < n$  for  $z = (\tilde{z}, 0)$ , and  $\frac{\partial f_n}{\partial z_j}(\tilde{z}, 0) = 0$ ,  $j = 1, \dots, n - 1$ , because  $f_n(\tilde{z}, 0) = 0$ . Therefore,  $\frac{\partial f_N}{\partial z_n}(\tilde{z}, 0) = 0$  for  $(\tilde{z}, 0) \in E_1$ . By continuity,

$$\frac{\partial f_N}{\partial z_n}(0) = 0, \quad (4)$$

which contradicts (3).

For the proof in the general case we will need the following technical result. We denote by  $\text{reg } E$  the locus of regular points of a complex analytic set  $E$ , i.e., the points near which  $E$  is locally a complex manifold. Then  $\text{sing } E = E \setminus \text{reg } E$  is the singular locus of  $E$ .

**Proposition 3.** *There exist a sequence of points  $\{p^\nu\} \subset \text{reg } E$  and two sequences of complex affine maps  $A^\nu : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $B^\nu : \mathbb{C}^N \rightarrow \mathbb{C}^N$  such that for every  $\nu = 1, 2, \dots$ , the following holds*

- (i)  $\text{rank}(f|_E) = n - 1$  at  $p^\nu$ , and  $f(p^\nu) \in \text{reg } E'$ .
- (ii)  $A^\nu(p^\nu) = p^\nu$  and  $B^\nu(f(p^\nu)) = f(p^\nu)$ .
- (iii) The transformations  $A^\nu$ ,  $B^\nu$  converge to the identity maps  $I_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and  $I_N : \mathbb{C}^N \rightarrow \mathbb{C}^N$  respectively.
- (iv)  $dA^\nu$  maps  $T_{p^\nu} E$  onto  $\{v \in T_{p^\nu} \mathbb{C}^n : v_n = 0\}$  and  $dB^\nu$  maps  $T_{f(p^\nu)} E'$  onto  $\{v \in T_{f(p^\nu)} \mathbb{C}^N : v_N = 0\}$ .

Theorem 1 can be easily deduced from Proposition 3. Indeed, consider the sequence of maps  $f^\nu = B^\nu \circ f \circ (A^\nu)^{-1}$ . The above arguments show that

$$\frac{\partial f_N^\nu}{\partial z_n}(p^\nu) \rightarrow 0 \quad \text{as } \nu \rightarrow \infty,$$

which yields (4). Again, we obtain a contradiction with the Hopf lemma.  $\square$

The rest of the section is devoted to the proof of Proposition 3. We will need the following

**Lemma 4.** *Let  $U \subset \mathbb{C}^n$  be a neighbourhood of the origin,  $M \ni 0$  be a real hypersurface in  $U$  with a defining function  $\rho \in C^1(U)$ ,*

$$\rho(z) = 2x_n + o(|z|). \quad (5)$$

*Let  $A \subset U$  be an analytic set of pure dimension  $d$ ,  $1 \leq d < n$ , such that  $0 \in A \subset \{z \in U : \rho(z) \geq 0\}$ . Then there exists an open subset  $V \subset \text{reg } A$  with  $0 \in \bar{V}$  such that for any point  $p \in V$  the tangent plane  $T_p A$  is contained in a complex hyperplane*

$$L_p = \{v \in \mathbb{C}^n : v_n = \sum_{k=1}^{n-1} a_k(p) v_k\},$$

*and  $\lim_{V \ni p \rightarrow 0} a_k(p) = 0$  for any  $k = 1, 2, \dots, n-1$ .*

*Proof.* Let  $C_0(A)$  be the tangent cone of  $A$  at 0. It is defined by  $C_0(A) = \lim_{t \rightarrow 0} A_t$ , where  $A_t = \{tz : z \in A\}$ ,  $t \in \mathbb{R}_+$ , are isotropic dilations of  $A$ . The set  $C_0(A)$  is a complex cone of dimension  $d$ , i.e., it is invariant under complex dilations  $z \rightarrow tz$ ,  $t \in \mathbb{C} \setminus \{0\}$  (see, e.g., [4]) and  $0 \in C_0(A) \subset \{z_n \geq 0\}$ . The last inclusion follows from  $A_t \subset \{z : t\rho(z/t) \geq 0\}$  and  $t\rho(z/t) \rightarrow 2x_n$  as  $t \rightarrow \infty$  because of (5). By the maximum principle we conclude that

$$C_0(A) \subset \{z_n = 0\}. \quad (6)$$

Since  $\dim C_0(A) = d$ , there exists a complex plane  $L \ni 0$ ,  $\dim L = n-d$ , such that  $L \cap C_0(A) = \{0\}$ . Without loss of generality we assume that

$$L = \{z \in \mathbb{C}^n : z_1 = 0, \dots, z_d = 0\}. \quad (7)$$

Let  $\tilde{z} = (z_1, \dots, z_d)$ ,  $\tilde{\tilde{z}} = (z_{d+1}, \dots, z_{n-1})$  so that  $z = (\tilde{z}, \tilde{\tilde{z}}, z_n)$ . It follows from (6) that  $|z_n| = o(|\tilde{z}| + |\tilde{\tilde{z}}|)$  on  $A$ , i.e., there exists a continuous function  $\alpha(t) \geq 0$  for  $t \geq 0$  such that

$$|z_n| \leq \alpha(|\tilde{z}| + |\tilde{\tilde{z}}|)(|\tilde{z}| + |\tilde{\tilde{z}}|), \quad z \in A. \quad (8)$$

We also have the following estimate for some  $c_1 > 0$  and all  $z \in C_0(A)$ :

$$|z_n| + |\tilde{\tilde{z}}| \leq c_1 |\tilde{z}|, \quad (9)$$

which follows from  $L \cap C_0(A) = \{0\}$  and (7). This implies that the origin is an isolated point of  $L \cap A$ . Hence, (9) also holds for  $z \in A$ , possibly with a different  $c_1$ .

Now we can choose

$$U = \tilde{U} \times \tilde{\tilde{U}} \times U_n \subset \mathbb{C}^d \times \mathbb{C}^{n-d-1} \times \mathbb{C}$$

such that  $\pi : A \cap U \rightarrow \tilde{U}$  is a branched analytic covering of some multiplicity  $m \geq 1$ . Its discriminant set  $\tilde{\sigma} \subset \tilde{U}$  and the tangent cone  $C_0(\tilde{\sigma}) \subset \mathbb{C}^d$  are analytic sets of dimension at most  $d-1$ . Therefore, there exists a complex line  $\tilde{l} \subset \mathbb{C}^d$  such that  $C_0(\tilde{\sigma}) \cap \tilde{l} = \{0\}$ . We may assume that  $\tilde{l} = \{(z_1, 0, \dots, 0) \in \mathbb{C}^d : z_1 \in \mathbb{C}\}$ . Since  $C_0(\tilde{\sigma})$  is a closed cone, there exists  $\delta > 0$  such that

$$\{\tilde{z} \in \mathbb{C}^d : |z_j| < \delta |z_1|, \quad j = 2, \dots, d\} \cap C_0(\tilde{\sigma}) = \emptyset. \quad (10)$$

With possibly smaller  $\delta > 0$  we also have

$$\{\tilde{z} \in \tilde{U} : |z_j| < \delta|z_1|, j = 2, \dots, d\} \cap \tilde{\sigma} = \emptyset. \quad (11)$$

The set

$$\tilde{V}_\delta := \{\tilde{z} \in \tilde{U} : |z_j| < \delta|z_1|, j = 2, \dots, d\} \cap \{\tilde{z} \in \tilde{U} : \operatorname{Re} z_1 > 0\}$$

is simply connected, open in  $\tilde{U}$  and contains the origin in its closure. Since  $\tilde{V}_\delta \cap \tilde{\sigma} = \emptyset$  the set  $A \cap (\tilde{V}_\delta \times \tilde{U} \times U_n)$  is the union of the graphs of  $m$  holomorphic mappings  $\tilde{V}_\delta \rightarrow \tilde{U} \times U_n$ . Consider one of them,  $H = (\tilde{h}, h_n)$ , and let  $A_\delta = A \cap (\tilde{V}_\delta \times \tilde{U} \times U_n)$  be its graph. For any  $p = (\tilde{p}, \tilde{p}, p_n) \in A_\delta$  the tangent plane  $T_p A$  is contained in the tangent plane at  $p$  to the hypersurface in  $\tilde{V}_\delta \times \tilde{U} \times U_n$  defined by one equation  $z_n = h_n(\tilde{z})$ , which is given by

$$\left\{ v \in \mathbb{C}^n : v_n = \sum_{k=1}^d a_k(\tilde{p}) v_k \right\}, \quad a_k(\tilde{p}) = \frac{\partial h_n}{\partial z_k}(\tilde{p}).$$

Thus, to finish the proof of the lemma, it is sufficient to show that

$$\lim_{\tilde{V}_\delta \ni \tilde{p} \rightarrow 0} \frac{\partial h_n}{\partial z_k}(\tilde{p}) = 0, \quad k = 1, \dots, d. \quad (12)$$

Using (8)–(11) we successively obtain for certain constants  $c_j > 0$  and all  $\tilde{p} \in \tilde{V}_\delta$ , with  $\delta \ll 1$ , the following estimates:

$$\begin{aligned} |p_1| &\leq |\tilde{p}| \leq c_1 |p_1|, \\ \operatorname{dist}(\tilde{p}, c_0(\tilde{\sigma})) &\geq c_2 |\tilde{p}| \geq c_2 |p_1|, \\ \operatorname{dist}(\tilde{p}, \tilde{\sigma}) &\geq c_3 |p_1|. \end{aligned}$$

If  $B(\tilde{p}, \tilde{\sigma})$  denotes the ball  $\{\tilde{z} \in \mathbb{C}^d : |\tilde{z} - \tilde{p}| < r\}$ , then  $B(\tilde{p}, c_4 |p_1|) \subset \tilde{V}_\delta$ , and  $|\tilde{z}| \leq c_5 |p_1|$  for all  $\tilde{z} \in B(\tilde{p}, c_4 |p_1|)$ . For  $z \in A$  with  $\tilde{z} \in B(\tilde{p}, c_4 |p_1|)$  we have

$$|h_n(\tilde{z})| = |z_n| \leq \alpha(|\tilde{z}| + |\tilde{z}|) (|\tilde{z}| + |\tilde{z}|) \leq c_5 \alpha(c_5 |\tilde{z}|) |\tilde{z}| \leq c_6 \alpha(c_6 \alpha |p_1|) |p_1|.$$

Now by the Schwarz lemma applied to  $h_n(\tilde{z})$  in  $B(\tilde{p}, c_4 |p_1|)$  we get

$$\left| \frac{\partial h_n}{\partial z_k}(\tilde{p}) \right| \leq c_7 \alpha(c_6 |p_1|),$$

and (12) follows from  $\lim_{t \rightarrow 0^+} \alpha(t) = 0$ . □

*Proof of Proposition 3.* The set

$$E_1 := \{z \in \operatorname{reg} E : \operatorname{rank}(f|_E) < n - 1 \text{ at } z\} \cup \operatorname{sing} E$$

is nowhere dense and closed in  $E$ . Therefore,  $E'_1 := f(E)$  is closed and nowhere dense in  $E'$ . By Lemma 4 with  $A = E'$  and  $M = \partial D'$  there exist a sequence  $p^\nu \in \operatorname{reg} E'$  and a sequence  $p^\nu \in \operatorname{reg} E$  such that

- (a)  $p^\nu = f(p^\nu)$ ,
- (b)  $\lim_\nu p^\nu = 0$ ,  $\lim_\nu p^\nu = 0'$ ,
- (c)  $\operatorname{rank}(f|_E) = n - 1$  at each  $p^\nu$ ,
- (d) for every  $\nu$ ,

$$T_{p^\nu} E' \subset \left\{ v \in \mathbb{C}^N : v'_N = \sum_{k=1}^{N-1} a'_{k\nu} v'_k \right\} \quad (13)$$

and

$$\lim_{\nu \rightarrow \infty} a'_{k\nu} = 0, \text{ for any } k = 1, \dots, N-1. \quad (14)$$

We claim that

$$T_{p^\nu} E \subset \left\{ v \in \mathbb{C}^n : v_n = \sum_{k=1}^{n-1} a_{k\nu} v_k \right\} \quad (15)$$

with

$$\lim_{\nu \rightarrow \infty} a_{k\nu} = 0, \quad k = 1, \dots, n-1. \quad (16)$$

Since  $f$  is holomorphic near the origin and sends  $\partial D$  into  $\partial D'$ , the last component  $f_N$  of  $f$  is of the form

$$f_N(z) = \mu z_N + o(|z|), \quad (17)$$

where  $\mu \neq 0$  by the Hopf lemma. The equations of  $T_{p^\nu}$  can be obtained from  $df_{p^\nu}(T_{p^\nu} E) \subset T_{p^\nu} E'$ . Using (13), (14), and (17) we conclude that  $T_{p^\nu} E$  are of the form (15) and the coefficients  $a_{k\nu}$  satisfy (16) because of (14) and (17). The transformations  $A^\nu$  and  $B^\nu$  can be defined by

$$\begin{aligned} A^\nu : (z_1, \dots, z_{n-1}, z_n) &\mapsto \left( z_1, \dots, z_{n-1}, z_n - \sum_{k=1}^{n-1} a_{k\nu} (z_k - p_k^\nu) \right), \\ B^\nu : (z'_1, \dots, z'_{N-1}, z'_N) &\mapsto \left( z'_1, \dots, z'_{N-1}, z'_N - \sum_{k=1}^{N_1} a'_{k\nu} (z'_k - p_k'^\nu) \right). \end{aligned}$$

They satisfy the required properties, and this completes the proof of the proposition and Theorem 1.  $\square$

#### 4. PROOF OF THEOREM 2

The crucial step in the proof of Theorem 2 is the following

**Lemma 5.** *If in the situation of Theorem 2 every irreducible component  $E \ni a$  of a branch locus of the correspondence that extends  $f$  enters  $D$  at  $a$ , then  $f$  extends holomorphically to  $a$ .*

*Proof.* Let  $U$  be a small neighbourhood of  $a$ , and  $F : U \rightarrow \mathbb{C}^n$  be the correspondence that extends the map  $f$  near the point  $a$ . Let  $E$  be the branch locus of  $F$  in  $U$ . Then  $E$  is a complex analytic set of pure dimension  $n-1$ . Since every component of  $E$  enters the domain  $D$  at  $a$ , we may choose the neighbourhood  $U$  so small that for every irreducible component  $\tilde{E}$  of  $E$ , the set  $\tilde{E} \cap D$  is nonempty and open in  $\tilde{E}$ .

Let  $S = E \setminus D$ . We claim that  $U \setminus S$  is simply connected. For the proof we will show that every nontrivial cycle in  $U \setminus E$  is null-homotopic in  $U \setminus S$ , from this simple connectivity of  $U \setminus S$  follows. By the classical van Kampen-Zariski Theorem, see, e.g., [15], the fundamental group of  $U \setminus E$  is generated by the cycles that generate the fundamental group of  $L \setminus (E \cap L)$ , where  $L$  is a complex line intersecting  $E$  transversely and avoiding singular points of  $E$ . Let  $\gamma$  be a generator of  $\pi_1(L \setminus (E \cap L))$ . Then  $\gamma$  is homotopic to a small circle in  $L$  around a point  $p$  of the intersection of  $L$  with an irreducible component  $\tilde{E}$  of  $E$ . Further, the point  $p$  is a regular point of  $\tilde{E}$ , and  $\gamma \cap E = \emptyset$ . Since the locus of regular points of  $\tilde{E}$  is connected and  $\tilde{E} \cap D$  contains an open subset of  $\tilde{E}$  by the assumptions of the lemma, we can move the cycle  $\gamma$  along the locus of smooth points of  $\tilde{E}$  avoiding points in  $E$  until  $\gamma$  is entirely contained in  $D$ . This means that  $\gamma$  is null-homotopic in  $U \setminus S$ , and hence the latter is simply connected.

We next show that the map  $f$  defined in  $D \cap U$  extends as a holomorphic map along any path in  $U \setminus S$ . Indeed, on  $U \setminus E$  the correspondence  $F$  splits into a finite collection of holomorphic mappings, the branches of  $F$ . Fix a point  $b \in (U \cap \partial D) \setminus E$ . Then one of the branches of the correspondence  $F$  at  $b$  gives the extension of the map  $f$  to a neighbourhood of  $b$ . Taking any path  $\gamma$  in  $U \setminus E$  which starts at  $b$  we obtain the extension of  $f$  along  $\gamma$  by choosing the appropriate branches of  $F$  over the points in  $\gamma$ . This gives analytic continuation of  $f$  in the complement of  $E$  in  $U$ . Suppose now that  $\gamma$  intersects  $E \cap D$ . Without loss of generality assume that  $\gamma$  terminates at a point  $c \in E \cap D$  and  $\gamma \setminus \{c\} \subset U \setminus E$ . The set  $S$  is closed and has simply connected complement in  $U$ , hence, any two paths in the complement of  $S$  are homotopically equivalent. In particular, this means that the path  $\gamma$  can be homotopically deformed avoiding the set  $S$  so that the deformation  $\tilde{\gamma}$  of  $\gamma$  connects the points  $b$  and  $c$  along the path that is entirely contained in  $D \setminus E$  (except the end points). Furthermore, we claim that this can be done in such a way that no curve in the deformation family intersects  $E$  (except the end point). Indeed, consider the cycle  $\gamma \circ \tilde{\gamma}^{-1}$  which we slightly deform so that it does not intersect  $E$  near the point  $c$ . If  $\gamma \circ \tilde{\gamma}^{-1}$  is null-homotopic in  $U \setminus E$ , then the claim is trivial. If  $\gamma \circ \tilde{\gamma}^{-1}$  is a nontrivial cycle in  $U \setminus E$ , then as in the proof of simple connectivity of  $U \setminus S$ , we may represent this cycle as a sum of “small” cycles around smooth points of  $E$ . We then move these small cycles along the regular locus of  $E$  until all of them are contained in  $D$  (again we used the fact that every component of  $E$  enters the domain  $D$ ). As a result we conclude by the Monodromy theorem that the analytic continuation of  $f$  along  $\gamma$  and  $\tilde{\gamma}$  defines the same analytic element near the point  $c$ . But since  $\tilde{\gamma}$  is contained in  $D$ , extension along  $\tilde{\gamma}$  simply gives the map  $f$  already defined at  $c$ . This gives analytic continuation of  $f$  along any path in  $U \setminus S$ , which is single-valued by the Monodromy theorem.

Finally, since every component of  $E$  enters the domain  $D$  at  $a$ , the set  $S$  is not a complex analytic subset of  $U$ , and hence it is a removable singularity for the extension of  $f$  in  $U \setminus S$ . This shows that  $f$  extends to  $a$  as a holomorphic map.  $\square$

*Proof of Theorem 2.* Choose normal coordinates near the points  $a$ ,  $f(a)$  and assume  $a = 0$ ,  $f(a) = 0'$ . By  $\rho$  and  $\rho'$  we denote local defining functions of  $D \cap U$ , and  $D' \cap U'$  respectively, of the form

$$\rho(z, \bar{z}) = 2x_2 + \sum_{|k|, |l| \geq 1} a_{kl}(y_n) \bar{z}^k \bar{z}^l, \quad (18)$$

$$\rho'(z', \bar{z}') = 2x'_2 + \sum_{|k|, |l| \geq 1} a'_{kl}(y'_n) \bar{z}'^k \bar{z}'^l, \quad (19)$$

Let  $\lambda : U \rightarrow \mathbb{C}^{N+1}$ ,  $\lambda' : U' \rightarrow \mathbb{C}^{N'+1}$  be the Segre maps of  $\partial D$  and  $\partial D'$  near  $0 \in U$  and  $0' \in U'$  respectively. It is convenient to denote their components by  $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_N)$ ,  $\lambda' = (\lambda'_0, \lambda'_1, \dots, \lambda'_N)$  so that in normal coordinates

$$(a) \lambda_0(z) = z_n, \quad (b) \lambda'_0(z') = z'_N. \quad (20)$$

We will need some results from [10] which can be summarized as follows.

**Proposition 6** (Diederich and Pinchuk, [10]). *Let  $F : U \rightarrow U'$  be the correspondence extending  $f : D \cap U \rightarrow D' \cap U'$ , where  $U \ni 0$ ,  $U' \ni 0'$  are small enough. Then*

- (i) *there exists a single-valued (even injective) map  $\phi : \lambda(U) \rightarrow \lambda'(U')$  such that the following diagram commutes.*

$$\begin{array}{ccc} \lambda(U) & \xrightarrow{\phi} & \lambda'(U') \\ \downarrow \lambda & & \downarrow \lambda' \\ U & \xrightarrow{F} & U'. \end{array} \quad (21)$$

*For the (multiple-valued) correspondence  $F$  this means that for any  $z \in U$ , it commutes with any value of  $F(z)$  (Cor. 4.2 and 5.5 in [10]);*

- (ii)  *$F(D \cap U) \subset D' \cap U'$ ,  $F(\partial D \cap U) \subset \partial D' \cap U'$ ,  $F(U \setminus D) \subset U' \setminus D'$  (Prop. 7.1);*  
 (iii) *The map  $\lambda' \circ F$  is single-valued and holomorphic in  $U$  with  $\lambda'_0 \circ F(z) = b(z)z_n$  and  $b(0) \neq 0$  (Prop. 7.2);*  
 (iv)  *$F : U \rightarrow U'$  is locally proper at the origin, i.e.,  $F^{-1}(0) = \{0\}$  and therefore,  $F^{-1}$  is also a holomorphic correspondence near  $0'$  (Thm 5.1).*

We will assume that  $b(z) \equiv 1$ . This can be achieved by an additional change of coordinates in  $U$ . Of course, these coordinates may no longer be normal. Instead we have

$$F_n(z) = f_n(z) = z_n. \quad (22)$$

Denote by  $\Omega'$  a neighbourhood of  $\lambda'(0')$  in  $\mathbb{C}^{N'+1}$ . We can choose the sets  $U \ni 0$ ,  $U' \ni 0'$ ,  $\Omega' \ni \lambda'(0')$  such that the mappings  $f : D \cap U \rightarrow D' \cap U'$  and  $\lambda' : U' \rightarrow \Omega'$  are proper holomorphic. Consider for  $M > 0$  the following open sets

$$\begin{aligned} D'_M &= \left\{ z' \in U' : 2x'_n + M \sum_{k=0}^{N'} |\lambda'_k(z')|^2 < 0 \right\}, \\ D_M &= \left\{ z \in U : 2x_n + M \sum_{k=0}^{N'} |\lambda'_k(F(z))|^2 < 0 \right\}. \end{aligned}$$

The boundaries  $\partial D'_M$ ,  $\partial D_M$  near  $0'$  and  $0$  respectively, are real analytic and pseudoconvex because of (20)(b), and of finite type because of properness of  $\lambda'$  and Proposition 6(iv).

We first prove Theorem 2 under an additional assumption that  $D'_M \cap U' \subset D' \cap U'$ . It follows from Proposition 6 and (22) that  $D_M \cap U \subset D \cap U$  and  $f : D_M \cap U \rightarrow D'_M \cap U'$  is a proper holomorphic map. This implies that for

$$\Omega'_M = \{w \in \Omega' : 2\operatorname{Re} w_0 + M|w|^2 < 0\},$$

the map  $\lambda' \circ f : D_M \cap U \rightarrow \Omega'_M$  is also proper holomorphic. By Proposition 6(iii), the map  $\lambda' \circ F = \lambda' \circ f$  extends holomorphically to a neighbourhood of  $0 \in U$ .

Let  $E' \subset U'$  be the critical set of  $\lambda' : U' \rightarrow \Omega'$  and  $S \subset U$  be the branch locus of  $F : U \rightarrow U'$ . By Proposition 6,  $F(S) \subset E'$ , moreover,  $F(S)$  is contained in the  $(n-1)$ -dimensional part of  $E'$ . By Theorem 1 any  $(n-1)$ -dimensional component of  $E'$  enters  $D' \cap U'$  at  $0'$ . By Proposition 6(ii), any irreducible component of  $S$  also enters  $D \cap U$  at  $0$ , and thus  $f$  extends holomorphically to  $0$  by Lemma 5. This completes the proof of Theorem 2 in the case  $D'_M \cap U' \subset D' \cap U'$ . However,  $D'_M \cap U'$  is not necessarily a subset of  $D' \cap U'$  and the general proof of Theorem 2 requires an additional (mainly technical) argument.



As in [11], consider for  $M > 1$  two families of open sets depending on  $\varepsilon \in (-\frac{1}{M}, 0]$ :

$$\begin{aligned} D'_{M\varepsilon} &= \left\{ z' \in U' : 2x'_n + M \sum_{k=0}^{N'} |\lambda'_k(z')|^2 < \varepsilon \right\}, \\ D_{M\varepsilon} &= \left\{ z \in U : 2x_n + M \sum_{k=0}^{N'} |\lambda'_k \circ F(z)|^2 < \varepsilon \right\}, \end{aligned}$$

These families are increasing for increasing  $\varepsilon$  and  $D'_{M0} = D'_M$ ,  $D_{M0} = D_M$ . The next proposition summarizes some results in [11].

**Proposition 7** (Diederich and Pinchuk, [11]).

- (a) *The sets  $D'_{M\varepsilon}$ ,  $D_{M\varepsilon}$  are pseudoconvex and their boundaries are of finite type at all points in  $U$ , respectively  $U'$ , where they are smooth real analytic.*
- (b)  *$D'_{M\varepsilon} \subset D' \cap U$  and  $D_{M\varepsilon} \subset D \cap U$  if  $\varepsilon \in (-\frac{1}{M}, 0]$  is close to  $-\frac{1}{M}$ .*
- (c) *For  $M > 0$  sufficiently large and any  $\varepsilon \in (-\frac{1}{M}, 0]$  the nonsmooth part of  $\partial D'_{M\varepsilon}$  is contained in  $D' \cap U'$  and the nonsmooth part of  $\partial D_{M\varepsilon}$  is contained in  $D \cap U'$ .*

To finish the proof of Theorem 2 consider

$$\Omega'_{M\varepsilon} = \left\{ w' \in \mathbb{C}^{N'+1} : 2u_0 + M|w'|^2 < \varepsilon \right\}.$$

If  $M, \varepsilon$  are chosen as in Proposition 7, then  $f : D_{M\varepsilon} \rightarrow D'_{M\varepsilon}$  and  $\lambda \circ f : D_{M\varepsilon} \rightarrow \Omega'_{M\varepsilon}$  are proper holomorphic maps. Consider the largest  $\varepsilon_0 \in (-\frac{1}{M}, 0]$  such that  $f$  extends to a proper holomorphic map  $\tilde{f} : D_{M\varepsilon} \rightarrow D'_{M\varepsilon_0}$ . By Proposition 7,  $\tilde{f}$  is holomorphic on the nonsmooth part of  $\partial D_{M\varepsilon_0}$ . Let us show that  $\tilde{f}$  extends holomorphically to any smooth real analytic boundary point  $a \in U$  of  $D_{M\varepsilon}$ . We only need to consider the case  $a \in S$ . Applying, as before, Theorem 1 to the map  $\lambda' : D'_{M\varepsilon_0} \rightarrow \Omega'_{M\varepsilon_0}$ , we conclude that any irreducible  $(n-1)$ -dimensional component  $E'_j \ni \tilde{f}(a)$  of the critical set  $E'$  of  $\lambda'$  enters  $D'_{M\varepsilon_0}$  at  $\tilde{f}(a)$ . By Proposition 6, any irreducible component  $S_j \ni a$  of  $S$  enters  $D_{M\varepsilon_0}$  at  $a$ . Thus, by Lemma 5,  $\tilde{f}$  extends holomorphically to any such  $a$ . This means that  $f$  extends holomorphically to a neighbourhood of the closure of  $D_{M\varepsilon_0}$  and  $\varepsilon_0 = 0$ . This completes the proof.  $\square$

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